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EVALUATION OF INTEGRALS OF THE TYPE

$$I_{(2i+1)/2}(b,\ -c^2,\ \tau) = c^i \int_0^\tau \frac{[\exp b(\theta-\tau) - c^2/\theta]}{\theta^{(2i+1)/2}}\,d\theta \quad (i = 0,\ 1,\ \ldots)$$

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A representation of integrals of the type $I_{(2i+1)/2}(b,\ -c^2,\tau)$ is obtained in the form of infinite series. Tables of integrals $I_{3/2}$ and $I_{1/2}$ are presented.

Integrals of the type $I_{(2i+1)/2}$ are met in solving a variety of thermal problems for a half-space with sources whose intensity depends exponentially on time. Thus, for example, in the simplest case of a half-plane with zero initial temperature and uniform boundary conditions of the third kind in the presence of sources of specific power $q_0 \exp(-m\tau)$, the solution of the heat conduction problem has the form

$$T = \frac{q_0}{mc\gamma}[1-\exp(-m\tau)] + \frac{q_0 h\sqrt{a}}{c\gamma\sqrt{\pi}(m+h^2a)}\left(I_{1/2} + \frac{h\sqrt{a}}{m} I_{3/2}\right), \tag{1}$$

where h is the relative heat transfer coefficient;

$$I_{1/2} = \int_0^\tau [\exp m(\theta-\tau) - x^2/4a\theta]\ \theta^{-1/2}\,d\theta; \tag{2}$$

$$I_{3/2} = (x/2\sqrt{a})\int_0^\tau [\exp m(\theta-\tau) - x^2/4a\theta]\,\theta^{-3/2}\,d\theta. \tag{3}$$

These integrals are usually evaluated by numerical integration methods.

In this paper we propose a representation of integrals of the type $I_{(2i+1)/2}(b,\ -c^2,\ \tau)$ in the form of infinite series, which simplifies the calculations appreciably. Moreover, tables of numerical values of $I_{3/2}$ and $I_{1/2}$ are presented.

It is known that

$$\exp(b\theta) = \sum_{n=0}^{\infty} b^n\theta^n/n!. \tag{4}$$

Whence

$$I_{3/2}(b,\ -c^2,\ \tau) = c\int_0^\tau [\exp b(\theta-\tau) - c^2/\theta]\,\theta^{-3/2}\,d\theta =$$

$$= c\exp(-b\tau)\left[J_1 + \frac{b}{1!}J_2 + \frac{b^2}{2!}J_3 + \frac{b^3}{3!}J_4 + \cdots\right], \tag{5}$$

where

$$J_1 = \int_0^\tau [\exp(-c^2/\theta)]\,\theta^{-3/2}\,d\theta,\quad J_2 = \int_0^\tau [\exp(-c^2/\theta)]\,\theta^{-1/2}\,d\theta,\quad J_3 = \int_0^\tau [\exp(-c^2/\theta)]\,\theta^{1/2}\,d\theta. \tag{6}$$

The integral

$$I_{1/2}(b,\ -c^2,\ \tau) = \int_0^\tau [\exp b(\theta-\tau) - c^2/\theta]\,\theta^{-1/2}\,d\theta =$$

$$= \exp(-b\tau)\left[J_2 + \frac{b}{1!}J_3 + \frac{b^2}{2!}J_4 + \frac{b^3}{3!}J_5 + \cdots\right]. \tag{7}$$

Here the integrals J have the same values as in (6).

It is easy to see that the integrals J are incomplete gamma functions $\Gamma(\alpha,\ x)$:

$$J_1 = \int_0^\tau \exp\left(-\frac{c^2}{\theta}\right)\theta^{-3/2}\,d\theta = \frac{1}{C}\int_{c^2/\tau}^{\infty} \exp(-t)\,t^{1/2-1}\,dt = \frac{1}{C}\,\Gamma\left(\frac{1}{2},\ \frac{c^2}{\tau}\right), \tag{8}$$

Numerical Values of Integrals $I_{3/2}(b, -c^2, \tau)$ and $I_{1/2}(b, -c^2, \tau)$

$c\sqrt{hr}$	b = 0.01/hr for τ, hr							
	12	24	72	120	240	480	600	720
0	1.5720 / 6.3997	1.3943 / 8.3710	0.8627 / 10.7562	0.5338 / 10.5383	0.1608 / 8.2852	0.0146 / 5.3202	0.0044 / 4.5817	0.0013 / 4.0788
0.2	1.4810 / 5.7890	1.3450 / 7.8231	0.8580 / 10.4120	0.5391 / 10.3237	0.1680 / 8.2194	0.0176 / 5.3138	0.0064 / 4.5795	0.0027 / 4.0780
0.6	1.2952 / 4.6783	1.2413 / 6.7881	0.8445 / 9.7307	0.5470 / 9.8891	0.1816 / 8.0795	0.0235 / 5.2973	0.0103 / 4.5728	0.0055 / 4.0747
1.0	1.1099 / 3.7165	1.1328 / 5.8382	0.8262 / 9.0622	0.5517 / 9.4494	0.1941 / 7.9292	0.0294 / 5.2761	0.0143 / 4.5630	0.0083 / 4.0691
1.5	0.8888 / 2.7185	0.9947 / 4.7744	0.7974 / 8.2499	0.5534 / 8.8965	0.2083 / 7.7278	0.0365 / 5.2432	0.0191 / 4.5463	0.0118 / 4.0591
2.0	0.6884 / 1.9319	0.8581 / 3.8484	0.7631 / 7.4692	0.5506 / 8.3441	0.2208 / 7.5132	0.0434 / 5.2032	0.0238 / 4.5248	0.0152 / 4.0456
3.0	0.3726 / 0.8911	0.6055 / 2.3907	0.6818 / 6.0219	0.5334 / 7.2578	0.2408 / 7.0505	0.0564 / 5.1033	0.0330 / 4.4679	0.0219 / 4.0085
5.0	0.0709 / 0.1287	0.2427 / 0.7646	0.4965 / 3.6615	0.4637 / 5.2511	0.2615 / 6.0378	0.0789 / 4.8310	0.0497 / 4.3016	0.0344 / 3.8955
7.0	0.0074 / 0.0107	0.0723 / 0.1876	0.3222 / 2.0340	0.3688 / 3.5817	0.2600 / 4.9880	0.0961 / 4.4793	0.0637 / 4.0739	0.0454 / 3.7353
10.0	0.0001 / 0.0001	0.0066 / 0.0135	0.1370 / 0.7083	0.2259 / 1.8077	0.2277 / 3.5107	0.1111 / 3.8514	0.0787 / 3.6430	0.0584 / 3.4215
20.0	0.0000 / 0.0000	0.0000 / 0.0000	0.0014 / 0.0044	0.0141 / 0.0709	0.0672 / 0.6383	0.0888 / 1.7122	0.0790 / 1.9542	0.0681 / 2.0737
50.0	0.0000 / 0.0000	0.0000 / 0.0000	0.0000 / 0.0000	0.0000 / 0.0000	0.0000 / 0.0000	0.0013 / 0.0113	0.0033 / 0.0359	0.0058 / 0.0763
	b = 0.02/hr							
0	1.3943 / 5.9192	1.0968 / 7.1901	0.4199 / 7.1739	0.1608 / 5.8585	0.0146 / 3.7620	0.0001 / 2.4282	0.0000 / 2.1403	0.0000 / 1.9364
0.2	1.3241 / 5.3754	1.0709 / 6.7565	0.4295 / 7.0040	0.1709 / 5.7922	0.0188 / 3.7553	0.0013 / 2.4280	0.0008 / 2.1401	0.0006 / 1.9363
0.6	1.1743 / 4.3754	1.0102 / 5.9233	0.4448 / 6.6539	0.1895 / 5.6479	0.0272 / 3.7368	0.0036 / 2.4260	0.0024 / 2.1389	0.0018 / 1.9354
1.0	1.0184 / 3.4981	0.9397 / 5.1428	0.4549 / 6.2937	0.2060 / 5.4895	0.0353 / 3.7119	0.0059 / 2.4222	0.0040 / 2.1363	0.0029 / 1.9335
1.5	0.8257 / 2.5768	0.8424 / 4.2511	0.4609 / 5.8352	0.2236 / 5.2745	0.0450 / 3.6717	0.0088 / 2.4148	0.0059 / 2.1314	0.0044 / 1.9299
2.0	0.6461 / 1.8423	0.7398 / 3.4598	0.4601 / 5.3742	0.2378 / 5.0435	0.0542 / 3.6220	0.0117 / 2.4045	0.0079 / 2.1245	0.0058 / 1.9248

(table continued)

c$\sqrt{hr}$	b = 0.02/hr for τ, hr							
	12	24	72	120	240	480	600	720
3.0	0.3552 / 0.8581	0.5372 / 2.1852	0.4410 / 4.4696	0.2565 / 4.5471	0.0710 / 3.4964	0.0172 / 2.3756	0.0117 / 2.1049	0.0086 / 1.9103
5.0	0.0689 / 0.1256	0.2238 / 0.7162	0.3554 / 2.8617	0.2596 / 3.5010	0.0966 / 3.1576	0.0275 / 2.2858	0.0189 / 2.0435	0.0141 / 1.8648
7.0	0.0073 / 0.0105	0.0683 / 0.1787	0.2473 / 1.6558	0.2292 / 2.5149	0.1108 / 2.7391	0.0362 / 2.1579	0.0253 / 1.9548	0.0190 / 1.7985
10.0	0.0001 / 0.0001	0.0064 / 0.0130	0.1129 / 0.6039	0.1564 / 1.3513	0.1125 / 2.0586	0.0455 / 1.9105	0.0328 / 1.7791	0.0251 / 1.6654
20.0	0.0000 / 0.0000	0.0000 / 0.0000	0.0013 / 0.0041	0.0117 / 0.0602	0.0436 / 0.4414	0.0440 / 0.9419	0.0375 / 1.0254	0.0318 / 1.0608
50.0	0.0000 / 0.0000	0.0000 / 0.0000	0.0000 / 0.0000	0.0000 / 0.0000	0.0000 / 0.0000	0.0009 / 0.0079	0.0021 / 0.0234	0.0035 / 0.0468
	b = 0.03/hr							
0	1.2366 / 5.4821	0.8627 / 6.2101	0.2044 / 5.0446	0.0484 / 3.7363	0.0013 / 2.3549	0.0000 / 1.5811	0.0000 / 1.4023	0.0000 / 1.2733
0.2	1.1841 / 4.9978	0.8536 / 5.8667	0.2173 / 4.9602	0.0566 / 3.7153	0.0038 / 2.3339	0.0007 / 1.5810	0.0005 / 1.4022	0.0004 / 1.2732
0.6	1.0654 / 4.0969	0.8242 / 5.1947	0.2401 / 4.7769	0.0723 / 3.6636	0.0086 / 2.3489	0.0021 / 1.5798	0.0015 / 1.4014	0.0011 / 1.2726
1.0	0.9352 / 3.2962	0.7821 / 4.5514	0.2590 / 4.5770	0.0868 / 3.5999	0.0134 / 2.3401	0.0036 / 1.5775	0.0025 / 1.3998	0.0019 / 1.2714
1.5	0.7678 / 2.4447	0.7161 / 3.8012	0.2770 / 4.3085	0.1034 / 3.5047	0.0192 / 2.3238	0.0053 / 1.5731	0.0037 / 1.3967	0.0028 / 1.2691
2.0	0.6070 / 1.7583	0.6402 / 3.1224	0.2891 / 4.0250	0.1179 / 3.3939	0.0249 / 2.3018	0.0071 / 1.5668	0.0049 / 1.3924	0.0037 / 1.2659
3.0	0.3388 / 0.8269	0.4783 / 2.0036	0.2971 / 3.4353	0.1408 / 3.1338	0.0355 / 2.2411	0.0105 / 1.5492	0.0074 / 1.3801	0.0055 / 1.2566
5.0	0.0670 / 0.1225	0.2068 / 0.6723	0.2632 / 2.2968	0.1618 / 2.5182	0.0532 / 2.0619	0.0169 / 1.4941	0.0119 / 1.3414	0.0090 / 1.2276
7.0	0.0071 / 0.0103	0.0646 / 0.1704	0.1948 / 1.3755	0.1550 / 1.8768	0.0648 / 1.8238	0.0224 / 1.4152	0.0160 / 1.2855	0.0121 / 1.1852
10.0	0.0001 / 0.0001	0.0062 / 0.0126	0.0946 / 0.5219	0.1150 / 1.0561	0.0702 / 1.4120	0.0285 / 1.2611	0.0208 / 1.1743	0.0161 / 1.1001
20.0	0.0000 / 0.0000	0.0000 / 0.0000	0.0012 / 0.0038	0.0100 / 0.0520	0.0314 / 0.3314	0.0288 / 0.6418	0.0244 / 0.6901	0.0207 / 0.7098
50.0	0.0000 / 0.0000	0.0000 / 0.0000	0.0000 / 0.0000	0.0000 / 0.0000	0.0000 / 0.0000	0.0007 / 0.0061	0.0015 / 0.0172	0.0024 / 0.0335

$$J_2 = \int_0^\tau \exp\left(-\frac{c^2}{\theta}\right)\theta^{-1/2}\,d\theta = C\Gamma\left(-\frac{1}{2},\ \frac{c^2}{\tau}\right),$$

$$J_3 = \int_0^\tau \exp\left(-\frac{c^2}{\theta}\right)\theta^{1/2}\,d\theta = C^3\Gamma\left(-\frac{3}{2},\ \frac{c^2}{\tau}\right), \tag{8 (cont'd)}$$

. . . .

But

$$\Gamma\left(\frac{1}{2},\ x\right) = \sqrt{\pi}\,\mathrm{erfc}(\sqrt{x}) \tag{9}$$

and

$$\Gamma(\alpha+1,\ x) = \alpha\Gamma(\alpha,\ x) + x^\alpha\exp(-x). \tag{10}$$

Using these relations, one can obtain a very simple type of algorithm for evaluating the integrals $I_{(2i+1)/2}(b, -c^2, \tau)$, namely:

$$I_{(2i+1)/2}(b,\ -c^2,\ \tau) = \left[K_{-1} + \sum_{n=0}^{\infty} K_n\right] \quad (i = 0,\ 1,\ 2,\ \ldots), \tag{11}$$

where

$$K_n = \frac{2}{(n+1)(2n+3-2i)}\,[D_n - bc^2K_{n-1}]; \tag{12}$$

$$K_{-1} = G^{-(2i-1)}\,\Gamma\left(\frac{2i-1}{2},\ \frac{c^2}{\tau}\right); \tag{13}$$

$$\left.\begin{array}{l} D_0 = \tau^{-\frac{2i-1}{2}}\exp(-c^2/\tau - q)\,q \\ \ldots\ldots\ldots\ldots \\ D_n = D_{n-1}\,q/n \end{array}\right\}; \tag{14}$$

$$q = b\tau;\ \Gamma\left(\frac{1}{2},\ x\right) = \sqrt{\pi}\,\mathrm{erfc}(\sqrt{x});\ \Gamma(\alpha+1,\ x) = \alpha\Gamma(\alpha,\ x) + x^\alpha\exp(-x).$$

Numerical values of the integrals $I_{3/2}(b,\ -c^2,\ \tau)$ and $I_{1/2}(b,\ -c^2,\ \tau)$ are given in the table. The calculations were based on algorithm (10) and (13) using a "Ural-2" computer.

Algorithm (10) and (13) is suitable for evaluating integrals of type $I_{(2i+1)/2}(b,\ -c^2,\ \tau)$ for any positive integer i. It is also easy to propose recurrence relations by means of which the integrals $I_{(2i+1)/2}$ may be evaluated for i > 1, using tabulated values of $I_{3/2}$ and $I_{1/2}$.

We find the derivative

$$\frac{d}{d\tau} = \frac{\exp(b\tau - c^2/\tau)}{\tau^{1/2}} = b\,\frac{\exp(b\tau - c^2/\tau)}{\tau^{1/2}} + c^2\,\frac{\exp(b\tau - c^2/\tau)}{\tau^{5/2}} - \frac{1}{2}\,\frac{\exp(b\tau - c^2/\tau)}{\tau^{3/2}}. \tag{15}$$

Integrating both sides of (14) between 0 and τ and multiplying the result by $\exp(-b\tau)$, we have

$$cI_{5/2} = c\tau^{-1/2}\exp\left(-\frac{c^2}{\tau}\right) - bcI_{1/2} + \frac{1}{2}\,I_{3/2}. \tag{16}$$

Similarly, it may be shown that there are recurrence relations for the integrals:

$$cI_{(2i+1)/2} = c^{i-1}\tau^{-(2i-3)/2}\exp(-c^2/\tau) - bcI_{(2i-3)/2} + [(2i-3)/2]\,I_{(2i-1)/2} \tag{17}$$

$$(i = 2,\ 3,\ \ldots).$$

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